Unit 2 Theory of Quadratic Equations

Q: Define discriminant.
The nature of the roots of quadratic equation depends on the value of the expression $b^2 - 4ac$, which is called the discriminant of the quadratic equation $ax^2 + bx + c = 0$.

Q: How we can find nature of the roots of a quadratic equation through discriminant?
Roots of the quadratic equation $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ and its discriminant $b^2 - 4ac$

- **(i)** If $b^2 - 4ac > 0$ and is a perfect square then the roots are **rational** (real) and **unequal**.
- **(ii)** If $b^2 - 4ac > 0$ and is not a perfect square then the roots are **irrational** (real) and **unequal**.
- **(iii)** If $b^2 - 4ac = 0$, then the roots are **rational** (real) and **equal**.
- **(iv)** If $b^2 - 4ac < 0$, then the roots are **imaginary** (complex conjugates) and **unequal**.

Q: Define sum and product of the roots of a quadratic equation.
The sum and the product of the roots of an equation $ax^2 + bx + c = 0$ are

$$S = \alpha + \beta = -\frac{b}{a}$$
$$P = \alpha\beta = \frac{c}{a}$$

Q: Define symmetric functions.
Symmetric functions are those functions in which the roots involved are such that the value of the expression involving them remain unaltered, when roots are interchanged. For example, if

$$\begin{aligned} f(\alpha,\beta) &= \alpha^2 + \beta^2, \quad then \\ f(\beta,\alpha) &= \beta^2 + \alpha^2 \\ &= \alpha^2 + \beta^2 \quad \because \beta^2 + \alpha^2 = \alpha^2 + \beta^2 \\ &= f(\alpha,\beta) \end{aligned}$$

Q: Define synthetic division.
Synthetic division is the process of finding the quotient and remainder, when a polynomial is divided by a linear polynomial.

Q: Define simultaneous equations.
A system of equation $f(x,y) = 0$ and $g(x,y)$ having a common solution is called a system of simultaneous equations.

For example $x + y = 1$ and $x^2 - y^2 = 1$ is system of simultaneous equations.

Q: Write the properties of cube root of unity.
The cube roots of unity are $1, \omega = \frac{-1+\sqrt{-3}}{2}$ and $\omega^2 = \frac{-1-\sqrt{-3}}{2}$. The cube roots of unity have following properties.

- **(i)** The product of cube roots of unity is one. $i.e$ $(1)(\omega)(\omega^2) = \omega^3 = 1$
- **(ii)** The sum of all cube roots of unity is zero. $i.e$ $1 + \omega + \omega^2 = 0$
- **(iii)** Each of complex cube roots of unity is reciprocal of the each other.
- **(iv)** Each of complex cube roots of unity is the square of the each other.

Q: Prove that the product of cube roots of unity is one.
The cube roots of unity are $1, \omega = \frac{-1+\sqrt{-3}}{2}$ and $\omega^2 = \frac{-1-\sqrt{-3}}{2}$.

$$The\ product\ of\ all\ roots = (1)(\omega)(\omega^2)$$

$$= (1)\left(\frac{-1+\sqrt{-3}}{2}\right)\left(\frac{-1-\sqrt{-3}}{2}\right)$$
$$= \frac{(-1)^2 - \left(\sqrt{-3}\right)^2}{4}$$
$$= \frac{1-(-3)}{4}$$
$$= \frac{1+3}{4}$$
$$= \frac{4}{4}$$
$$The\ product\ of\ all\ roots = 1$$

Thus, $(1)(\omega)(\omega^2) = \omega^3 = 1$

Q: Prove that the sum of all cube roots of unity is zero.

The cube roots of unity are $1, \omega = \frac{-1+\sqrt{-3}}{2}$ and $\omega^2 = \frac{-1-\sqrt{-3}}{2}$.

$$The\ sum\ of\ all\ roots = 1 + \omega + \omega^2$$
$$The\ sum\ of\ all\ roots = 1 + \frac{-1+\sqrt{-3}}{2} + \frac{-1-\sqrt{-3}}{2}$$
$$The\ sum\ of\ all\ roots = \frac{2 + \left(-1+\sqrt{-3}\right) + \left(-1-\sqrt{-3}\right)}{2}$$
$$The\ sum\ of\ all\ roots = \frac{2 - 1 + \sqrt{-3} - 1 - \sqrt{-3}}{2}$$
$$The\ sum\ of\ all\ roots = \frac{0}{2}$$
$$The\ sum\ of\ all\ roots = 0$$

Thus, $1 + \omega + \omega^2 = 0$

Q: Prove that each of complex cube roots of unity is reciprocal of the each other.

The complex cube roots of unity are $\omega = \frac{-1+\sqrt{-3}}{2}$ and $\omega^2 = \frac{-1-\sqrt{-3}}{2}$. We know that

$$\omega^3 = 1$$
$$\Rightarrow \omega.\omega^2 = 1$$

So

$$\omega^2 = \frac{1}{\omega} \qquad OR \qquad \omega = \frac{1}{\omega^2}$$

Q: Prove that each of complex cube roots of unity is the square of the each other.

The complex cube roots of unity are $\omega = \frac{-1+\sqrt{-3}}{2}$ and $\omega^2 = \frac{-1-\sqrt{-3}}{2}$. We have to prove that

(i) $\left(\frac{-1+\sqrt{-3}}{2}\right)^2 = \frac{-1-\sqrt{-3}}{2}$ and

(ii) $\left(\frac{-1-\sqrt{-3}}{2}\right)^2 = \frac{-1+\sqrt{-3}}{2}$

Proof:

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^2 = \frac{(-1)^2 + \left(\sqrt{-3}\right)^2 + 2(-1)\left(\sqrt{-3}\right)}{4}$$
$$= \frac{1 + (-3) - 2\sqrt{-3}}{4}$$
$$= \frac{1 - 3 - 2\sqrt{-3}}{4}$$
$$= \frac{-2 - 2\sqrt{-3}}{4}$$
$$= \frac{2\left(-1 - \sqrt{-3}\right)}{4}$$
$$= \frac{-1 - \sqrt{-3}}{2}$$

$$\left(\frac{-1-\sqrt{-3}}{2}\right)^2 = \frac{(-1)^2 + \left(\sqrt{-3}\right)^2 - 2(-1)\left(\sqrt{-3}\right)}{4}$$
$$= \frac{1 + (-3) + 2\sqrt{-3}}{4}$$
$$= \frac{1 - 3 + 2\sqrt{-3}}{4}$$
$$= \frac{-2 + 2\sqrt{-3}}{4}$$
$$= \frac{2\left(-1 + \sqrt{-3}\right)}{4}$$
$$= \frac{-1 + \sqrt{-3}}{2}$$

Thus, if $= \frac{-1+\sqrt{-3}}{2}$, then $\omega^2 = \frac{-1-\sqrt{-3}}{2}$ and if $\omega = \frac{-1-\sqrt{-3}}{2}$, then $\omega^2 = \frac{-1+\sqrt{-3}}{2}$.

Q: How to write a quadratic equation given the sum and product of the roots?

If α and β are roots of equation, then

$$x^2 - (sum\ of\ the\ roots)x + product\ of\ the\ roots = 0$$
$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$